Vortex Stability and Breakdown: Survey and Extension

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I. Introduction

VORTEX cores are fluid-mechanical waveguides capable of supporting dispersive waves, but complex instabilities may be engendered in them under conditions often realized in practical situations. In addition, as shown by Hall,[1] substantial swirl velocities cause pressure gradients imposed at the edge of the vortex core to be amplified on the centerline, making the flow there highly sensitive to the environment in which the vortex resides. This sensitivity can result in large amplitude transitions of the flow structure, often leading to the formation of internal stagnation points and regions of reversed axial flow. Large amplitude disturbances result in the rapid decelerations known as "vortex breakdowns." The term as used here will refer to those disturbances leading to the development (in some frame of reference) of a stagnation point and flow reversal in a limited region. The phenomenon first attracted serious scientific interest when it was discovered to occur in the vortices formed above lifting surfaces with highly swept leading edges,[2] causing changes in the slopes of the lift, drag, and moment curves.[3,4]

This survey deals with the stability and breakdown of isolated incompressible vortices at high Reynolds numbers; instabilities arising from interactions between two or more vortices are not considered. The juxtaposition of the words "vortex stability" and "breakdown" in the title of this paper does not imply a causal relationship between the two varieties of events. A vortex can be unstable without breakdown, and it appears that axial flow reversals can be created in swirling flows without any sign of hydrodynamic instability to infinitesimal disturbances. A breakdown event causes significant modifications of the vortex structure, however, and experiments at high Reynolds numbers reveal that the flow downstream of it is always more unstable than that upstream. Breakdown acts as a switch that marks the onset of transition to a turbulent flow downstream if the flow upstream is laminar, or transition to a flow with a higher level of turbulent fluctuations downstream if the vortex upstream of breakdown is already turbulent.

With the possible exception of confined vortices at fairly low Reynolds numbers, the flow downstream of breakdown seems always to contain fluctuations that are not axially symmetric, even if the flow upstream is axially symmetric to a high degree of accuracy. The expansion of the vortex core in the wake of a breakdown event is due to the mixing associated with the instabilities and turbulence. Thus, whether instabilities are responsible for the existence of breakdown or not, they play an essential role in shaping the global structure of the flow and, therefore, in determining the aerodynamic effects of breakdown.

An appreciation of the kinds of experimental arrangements that have been used to study the vortex breakdown phenomenon helps to place the experimental work into context. This survey, therefore, begins in Sec. II with a brief discussion of the methods of formation of the vortices studied in the laboratory and arising in some cases of practical interest. This is followed in Sec. III by descriptions of the two major types of vortex breakdown: the "bubble," which will be called the B-type, or B-breakdown, and the "spiral" breakdown, which will be called the S-type, or S-breakdown. Both types are shown in Fig. 1, a much reproduced photograph taken from Lambourne and Bryer[5] of both types appearing simultaneously in leading-edge vortices above a delta wing, and in the photographs of Fig. 2 showing both forms in a tube. The question of the departure from axial symmetry and whether or not it can be ignored is taken up in a preliminary way in Sec. IV.

Criteria for breakdown based on theoretical arguments that neglect all nonaxisymmetric phenomena are considered in Sec. V. The main unifying element, whether originally recognized or not, in all formulations of criteria for onset of breakdown in purely axisymmetric theories is the concept of critical flow. This condition was first introduced by Squire[6] in his wave criterion for breakdown, enlarged and reinterpreted by Benjamin[7] in his theory of finite transitions between "conjugate" flow states, and shown by Ludwieg[8] and by Hall[9] to be common not only to the wave-based theories of Squire and of Benjamin, but also to earlier theories based upon ideas related to boundary-layer separation. The finite amplitude wave theories of Benjamin[10] and Leibovich,[11] and the transcritical theory of Leibovich and Randall[12] and Randall and Leibovich[13] are also centered around the criticality classification. The transcritical wave theory is outlined here, since the author believes that it contains the germ of a comprehensive description of the phenomena; it serves as a useful vehicle to present speculations, to be found in Sec. VII, on what a comprehensive theoretical description may turn out to be like.

The stability or instability of vortex cores at infinite Reynolds numbers is considered in Sec. VI partly from a theoretical point of view, and partly by reference to recent

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experiments. Ludwieg[14-16] advanced an explanation of vortex breakdown based upon hydrodynamic instability to three-dimensional disturbances. This has not won acceptance, since the flows upstream are stable. Nevertheless, it appears as though instability may well be important in fixing the location of breakdown, and, of course, the properties of the wake. Instability to three-dimensional disturbances plays an important part in the conceptual framework for breakdown suggested in Sec. VII, which arises from the analysis of experimental data summarized here.

Reviews of vortex breakdown are given by Hall[1,9] and Leibovich.[17] The reader should find them helpful; a recent general discussion by Legendre[18] will also be of interest.

II. Vortex Generation

To address questions concerning modifications of the flow structure in vortices as some control parameter, such as angle of incidence, is varied, it is necessary to begin with some model of the vortex which may be expected to lead to a tractable analysis. This requires that the "vortex" be isolated from its surroundings in some consistent way. All models so abstracted for the purpose of analysis of stability or of breakdown have been axially symmetric. This process of isolation leads to questions whose answers are not self-evident, particularly concerning the appropriate boundary conditions to impose on disturbances to the basic vortex flow model.

The problem of vortex breakdown first attracted serious scientific attention in the context of leading-edge vortices on highly swept leading edges, and the main aerodynamic interest probably continues to be in flows of this kind, or in trailing vortices; both are formed by the roll up of a vortex sheet. Early on, it was recognized that detailed experimental data was in many ways more easily obtained on vortices confined to tubes, and the bulk of available data are for such cases.

Figure 3 is a sketch of a leading-edge vortex, and Figs. 4 and 5 contain sketches of the two types of laboratory apparatus that have been used to study vortices confined to tubes. Each of these will be discussed in general terms.

Edge and Trailing Vortices

Experimental observation and measurement indicates that the vorticity in the separating boundary layer may be regarded as a vortex sheet as it leaves the separation line. As the sheet rolls up, its intersection with a plane normal to the lifting surface forms a spiral and the velocity profiles in the inner portion of the spiral appear to be nearly axially symmetric. This axisymmetric "core" grows in radius in the downstream direction, and occupies the interior of a conical region. The circulation about this core increases, essentially linearly with distance along the axis, as vorticity is continuously wound into the core region. Axial speeds on the centerline increase with distance as well, due to the continuous drop in centerline pressure as the core vorticity increases.

Outside of the conical core, the flow is neither irrotational nor axially symmetric; in fact, the definition of the core is somewhat arbitrary. In their discussion of the incompletely rolled up trailing vortex, for instance, Moore and Saffman[19] suggest a definition based upon the ratio of the distance between neighboring turns of the sheet to the distance from the center, using Kaden's[20] similarity model of roll up. The core is considered to be axisymmetric and the vorticity—which in the parent model is confined to sheets—is considered to be spread continuously throughout the region where this distance ratio remains less than an arbitrarily assigned value. In any event, the main requirement in isolating a vortical core presumably should be that the region abstracted contains most of the vorticity. Fortunately, it appears possible to define a core that is at the same time nearly axisymmetric, and this is a major simplifying feature. Its adoption, for example, permitted Hall[21] and Stewartson and Hall[22] to construct a successful theory for the leading-edge vortex at high Reynolds number. The theory represents the flow as rotational, inviscid, and conical except in a very slender, nonconical, viscous subcore.

It is important to note that the size of a vortical core formed by roll up of a vortex sheet is independent of Reynolds number (although the radius of the viscous subcore decreases with increasing Reynolds number). The center of a leading-edge vortex is generally located on the order of a core diameter above the lifting surface, and the core diameter may not be negligibly small compared to the local semispan of the lifting surface. Should the core diameter increase by a substantial fraction, as it does in a vortex breakdown, there may be significant aerodynamic consequences.

Sufficiently far downstream of a lifting surface, its associated vortex sheet may be considered to be completely rolled up into two trailing vortices, each of constant circulation. The core of each is then naturally definable. If diffusion and mutual interactions are ignored and each trailing vortex is treated as if the other were not there, then the

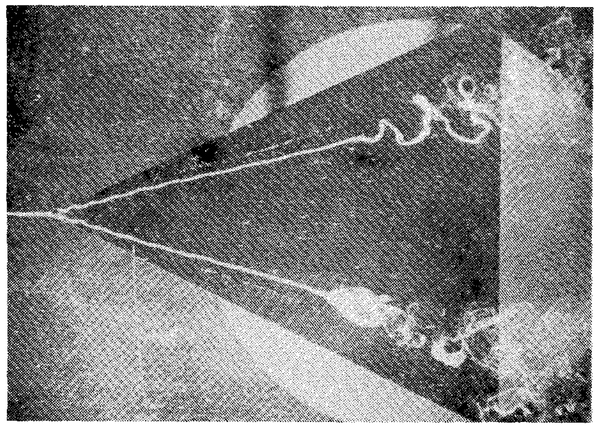

Fig. 1 Vortex breakdown in leading-edge vortices over a model delta wing, from Ref. 5. The breakdown in the upper portion of the figure is a spiral, or S-type; the one in the lower portion is a bubble, or B-type.

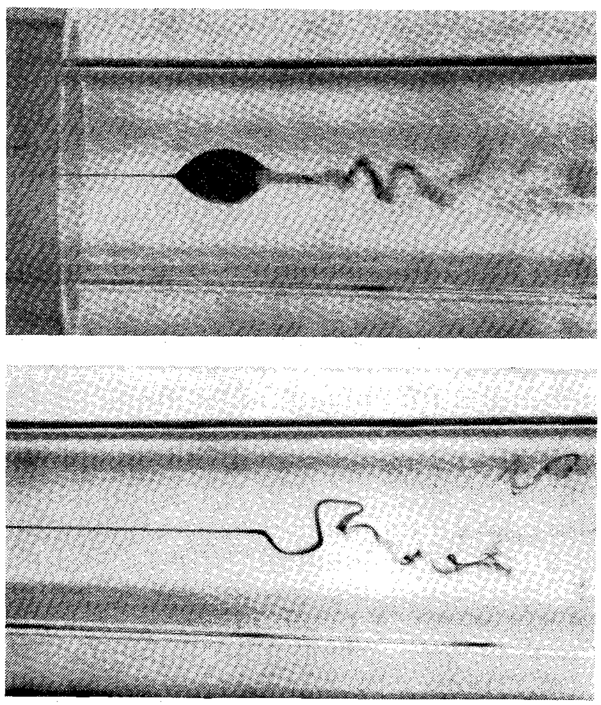

Fig. 2 Photographs of vortex breakdowns in a tube, from Ref. 34. a) B-type followed by S-breakdown downstream and b) S-type.

velocity field may be treated as columnar, with each streamline being a helix, and the flow outside the core as irrotational.

Vortex Generators with Tangential Jet Entry

Apparatus in which vortices are generated by jets directed tangent to the walls of a cylindrical cavity have been used in some laboratory experiments on vortex breakdown.[23-26] At large Reynolds number, apparatus of this design, and in particular the slot entries used by Escudier's group, produce vortex cores with vorticity deriving mainly from the roll up of a shear layer shed at the slot exit, as illustrated in Fig. 4. Vorticity shed from separation of the end wall boundary layer is also entrained in the core, but this is probably a smaller effect except near the end wall itself. Thus, some similarity in the structures of vortices produced this way to edge vortices in external aerodynamic flows is to be expected.

Velocity profiles measured by Escudier et al.[26] are shown in Fig. 6, alongside the data taken by Earnshaw[27] (figure shown is taken from Hall[21]) in two mutually perpendicular traverses through a leading-edge vortex over a delta wing model. Qualitative similarities are evident. In particular, in both sets of measurements, the azimuthal velocity component is nearly flat beyond the point of maximum swirling speed, although the swirl level in this flat stretch is larger for the leading-edge vortex. The axial velocity component is associated with azimuthal vorticity throughout the flow region shown: there is no external irrotational flow.

Escudier et al.[26] found that the following model can be fitted to their data for the swirl component:

$$V(r) = (\Gamma_c/2\pi r)\,[1-\exp(-r^2/r_0^2)] + \omega r/2 \qquad (1)$$

where Γ_c, r_0, and ω are constants for a profile at a fixed axial station; their profile constants have not been published, and no analytical fit has been suggested for the axial velocity component.

After leaving the vortex generation section, the flow enters a cylindrical exit tube. Presumably, the roll-up continues for at least some distance into the exit tube, but except for a sink of axial vorticity at the wall, all of the vorticity in the exit tube is convected from the swirl generator upstream. The core size in these vortices prior to any breakdown which may take place seems to be fixed by the roll-up process in the vortex generation section and varies weakly, if at all, with Reynolds number.

Vane Vortex Generators

The first experiments on vortex breakdown in tubes were conducted by Harvey,[28] and one of the main motivations was the removal of the asymmetries characteristic of leading-edge vortices. Swirl was produced in the apparatus, previously used by Titchener and Taylor-Russell[29] for another purpose, by means of a radial inflow through a set of vanes. After acquiring swirl in passing the vanes, the radial flow is turned through an annular channel created by an outer curved wall and a centerbody and exits axially into a cylindrical tube which serves as the test section. Since the flow accelerates through the turning section, the point of detachment is fixed at the tip of the centerbody. Sarpkaya[30-32] conducted experiments in a similar apparatus, but with conical test sections of generally small cone angles. Bellamy-Knights,[33] Faler and Leibovich,[34,35] and Garg and Leibovich[36] used apparatus modeled after Sarpkaya's.

Except for boundary layers on the centerbody and on the outer boundary, the flow in the vortex generator is essentially irrotational. In the test section, the flow is accurately irrotational in an annular region lying between a well-defined vortical core surrounding the tube centerline and the boundary layer on the outer wall. The vorticity in the core all derives from the boundary layer shed from the centerbody, and the circulation is constant for all circuits between the core and the wall boundary layer. The development of the vortex core by shedding of the centerbody boundary layer leads to a low centerline pressure and consequent formation of an axial jet just downstream that is confined to the vortex core.

Experimental data is well fitted by the analytical profiles for axial velocity W and swirl velocity V,

$$W = W_1 + W_2\exp(-\alpha r^2) \qquad V = \lambda[1-\exp(-\alpha r^2)]/r \quad (2)$$

where W_1, W_2, α, and λ are constants; these constants have been published for a number of experiments.[36]

In this type of apparatus, the size of the core is determined by the thickness of the boundary layer shed by the centerbody, and decreases with an increase in Reynolds number. This

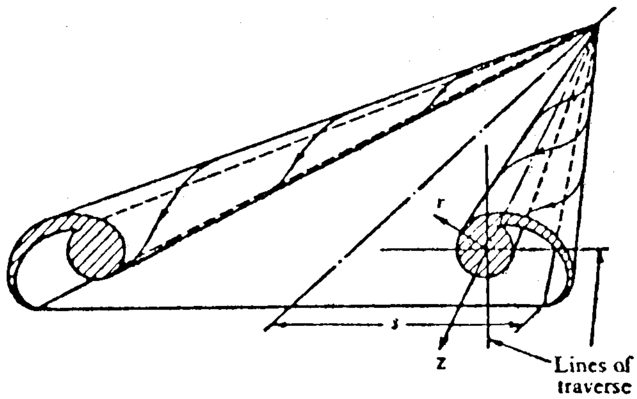

Fig. 3 Sketch of a leading-edge vortex, taken from Ref. 21.

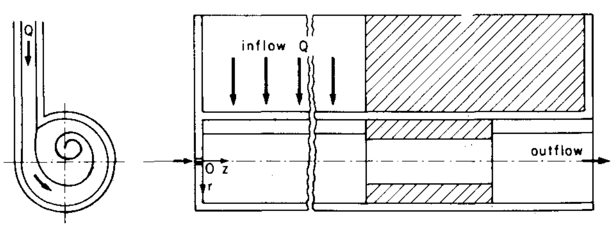

Fig. 4 Tangential slot-entry swirl generation apparatus used in Refs. 23-26. The rolling up of the shed vorticity layer is sketched as a spiral in the end-view on the left.

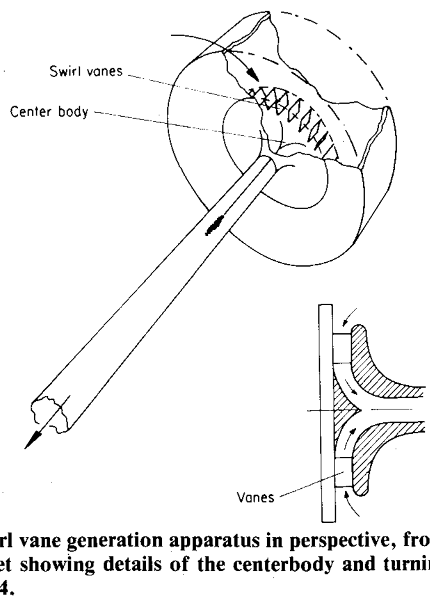

Fig. 5 Swirl vane generation apparatus in perspective, from Ref. 28, with an inset showing details of the centerbody and turning section, from Ref. 34.

core-size dependence leads to a complicated coupling between the two control variables, flow rate (Reynolds number) and vane angle; an increase of either parameter leads to an increase in core vorticity.

III. Vortex Breakdown: Descriptive Survey and Remarks

At large Reynolds numbers, breakdown assumes one of two characteristic geometric forms. In either form, a stagnation point appears on the vortex axis, followed by a limited region of reversed axial flow near the vortex centerline. A tracer released on the vortex axis upstream shows the B-breakdown to look much like a body of revolution placed in the flow; in the case of an S-breakdown, the tracer assumes a spiral or corkscrew shape that rotates about the vortex axis in the same sense as does fluid in the upstream vortex. In both forms the upstream approach flow has a jet-like axial velocity component with a centerline speed exceeding the axial velocity outside the core by a factor ranging up to three or even more. Deceleration of this strong axial jet to rest at an internal stagnation point is accomplished in an axial distance of one or two diameters of the upstream vortex core. Upstream of the deceleration interval, the presence of the breakdown is not evident to the eye, or in velocity measurements.

The stagnation point is followed by an axial interval of a very complicated nature which contains one or more regions of flow reversal. The bubble and the coherent corkscrew form are contained within a volume of whose radial extent is close to that of the upstream core radius, and both flow structures exhibit time-periodic velocity fields that are not axially symmetric. This zone typically has an axial extent of one to three diameters of the upstream vortex core. The flow downstream of the breakdown zone is wake-like and invariably turbulent with strong coherent periodic oscillations. The mean axial velocity deficit in this zone is reminiscent of the wake of a solid body of revolution, and the vortex core is greatly expanded.

Axial gradients in the approach flow upstream of the breakdown zone and in the wake downstream of it appear to be determined by globally imposed conditions, such as an overall (axial) pressure gradient, or the geometry of the container walls, or, if impressed forcing conditions are absent, by viscous decay. The axial scales of impressed forcing typically are much larger than the diameter of the vortex core; on these scales, the breakdown zone looks like a thin transition region joining the approach flow and the wake.

Measured values of the core expansion ratio (wake core diameter/approach flow core diameter) analyzed by Leibovich[17] (see Table 1 of Sec. V), using data from Garg and Leibovich,[36] range from slightly over 1.5 to nearly 3. Expansions associated with the B-form were found to be significantly larger than those associated with the S-form.

Most of the information that is available concerning breakdown, including the very important qualitative information concerning onset, position, and type, have been obtained by flow visualization using smoke or dye as a tracer. In particular, the designations of different types of breakdown, the B- and S-forms, were made on the basis of their appearances in flow visualization experiments. Are these two forms really different, or just visual variants of the same flow state? Recognizing that care must be taken in the interpretation of streakline data, it might not be prudent to infer from the distinctive appearances of the B-form and the S-form as revealed by smoke or dye that these actually mark distinct flow disturbances. It might be argued, for example, that there are minor wiggles of the upstream dye or smoke feeder due to very slight disturbances near the stagnation point that are sometimes large enough to periodically deflect the tracers, leading to what we see as the S-form, while at other times they are too weak to have a noticeable effect, and we identify the disturbance as a B-form. This argument, while plausible based only on the shapes of the dye filament, is not correct; at least two pieces of evidence exist to show that the two forms are in fact distinct flow disturbances. One piece of evidence, the different core expansion ratios, has already been discussed. The second has to do with the upstream flow conditions which lead to either the B- or the S-form.

Consider the flow in a conical tube apparatus of the Sarpkaya design at fixed flow rate (Reynolds number). For low values of swirl (i.e., small vane angle), the flow exhibits no unusual behavior. When the swirling speeds exceed a threshold value, a large amplitude disturbance form appears in the test section. For a Reynolds number based on tube diameter of, say, 3000 or larger, the breakdown that first appears is of S-type, according to Sarpkaya,[30] Faler and Leibovich,[34] and Garg and Leibovich.[36] If the swirl level is further increased, the breakdown moves to a position farther upstream. Although there is vacillation in position, the breakdown appears to have a definite mean location that depends upon the swirl. As the swirl is further increased this process continues until, at a second critical value of the swirl, the S-form transforms into a B-form, and moves rapidly to a

Table 1 Data from Refs. 17 and 36 with values of Froude number N appended

	Reynolds number breakdown type	A[a] or W[b]	δ	q	N	Core expansion ratio
a)	11,480[c] (1,920)[d]	A	0.89	1.98	1.86	1.6
	S[e]	W	−0.72	1.45	0.60	
b)	11,480 (1,920)	A	1.58	1.59	1.32	2.5
	B[f]	W	−0.94	0.97	0.44	
c)	14,100 (2,812)	A	0.97	1.86	1.47	1.8
	S	W	−0.94	1.43	0.57	
	14,100 (2,812)	A	1.4	1.62	1.37	2.0
	B	W	−0.87	1.18	0.45	
	20,160 (3,348)	A	1.04	1.85	1.42	2.1
	S	W	−0.82	1.38	0.46	
	20,160 (3,348)	A	1.48	1.64	1.32	2.8
	B	W	−0.91	1.04	0.45	

[a]Refers to data in approach flow. [b]Refers to data in wake flow. [c]First Reynolds number is based on tube radius. [d]Reynolds number in parentheses is based on core diameter. [e]Implies transition via bubble breakdown. [f]Implies transition via spiral breakdown.

new equilibrium location several core diameters upstream of the previous mean position. (The metamorphosis is described by Sarpkaya[30] and by Faler and Leibovich.[34]) Thus, the transition to a B-form is associated with a discontinuity in the mean position vs swirl-level functional relationship, and this discontinuity is the clearest evidence that the two disturbance forms are truly distinct.

There are borderline ranges of swirl levels for which the B-form and the S-form spontaneously transform one into the other, each transition causing the location of the newly established breakdown to move rapidly to the mean axial location appropriate to its type. The response of a B-form to increases of swirl is similar to the S-form, the mean location moves upstream; furthermore, the spontaneous reversion to an S-form becomes increasingly rare and, as the swirl is increased beyond some level, this reversion no longer is observed to occur.

The location of breakdown has been measured as a function of volume flow rate Q (or a Reynolds number based upon Q) and swirl, measured in dimensionless form by the parameter $\Omega = \pi\Gamma D/4Q$ originally introduced by Sarpkaya.[30] Here Γ is the circulation about the vortex core, a quantity that can be inferred from the inlet data (for example, the vane angle and placement in a vane swirl generator), and D is a characteristic diameter of the apparatus. Measurements of this kind are presented by Sarpkaya,[30-32] Faler and Leibovich,[34] and Escudier and Zehnder[25]; a limited sample is also given by Garg and Leibovich.[36] Except for Escudier and Zehnder, who use a tangential jet swirl device, the experiments cited use vane swirl generation, and a typical compilation of results is shown in Fig. 7. The dependence of the breakdown location with Re in these flows is probably not due to a viscous role in the breakdown mechanism, which appears to be essentially inviscid, and, although the point continues to be misinterpreted (cf. Escudier and Zehnder, Ref. 25, p. 118), it certainly cannot be attributed to viscous wall effects (the discussion here excludes phenomena at Reynolds numbers so low as to allow frictional influences of the wall to penetrate to the vortex core). Rather, the breakdown location at fixed Ω and increasing (decreasing) Re varies primarily because of the increase (decrease) in peak swirl and core vorticity associated with a reduction (enlargement) in core diameter; recall that a vane-centerbody apparatus produces a vortex with a core diameter proportional to $Re^{-1/2}$.

Escudier and Zehnder[25] show that the simple rule $Re\Omega^3 R = \text{const}$, where R is a dimensionless parameter associated with the swirl generation section, correlates conditions for the occurrence of vortex breakdown at a fixed axial location in their apparatus. They also suggest that this result is applicable to other experimental data. For data apparatus using vane swirl generation, the correlation reduces to $Re\Omega^2 = \text{const}$. Unfortunately, neither this result, nor any other functional relationship between Re and Ω, can possibly represent a universal criterion for the occurrence of breakdown at a fixed location, as Fig. 7 indicates, simply because B- and S-breakdowns have distinctly different locations as a function of Re and Ω in vane devices. In fact, as pointed out by Faler and Leibovich[34] and by Leibovich,[17] the relevant dynamics in vortices scale with the vortex core diameter, and that it is likely to be more revealing to describe data using a Reynolds number based on that length.

The main characteristics of vortex breakdowns above delta wings at high incidences appear to agree with the findings described above for flows in tubes, although detailed structural information of the breakdowns that occur in the former cases is not available. Nevertheless, the appearance of the two main types, the B-form and the S-form are both well documented,[5,37] with the S-form predominating. Since this is the low-swirl form of breakdown, this may merely imply that under conditions of aerodynamic interest, the swirl level is not high enough to render the B-breakdown the stable form. As the incidence is increased, and thereby the swirl level, an existing breakdown moves upstream like its tube counterpart.[3,4,37,38]

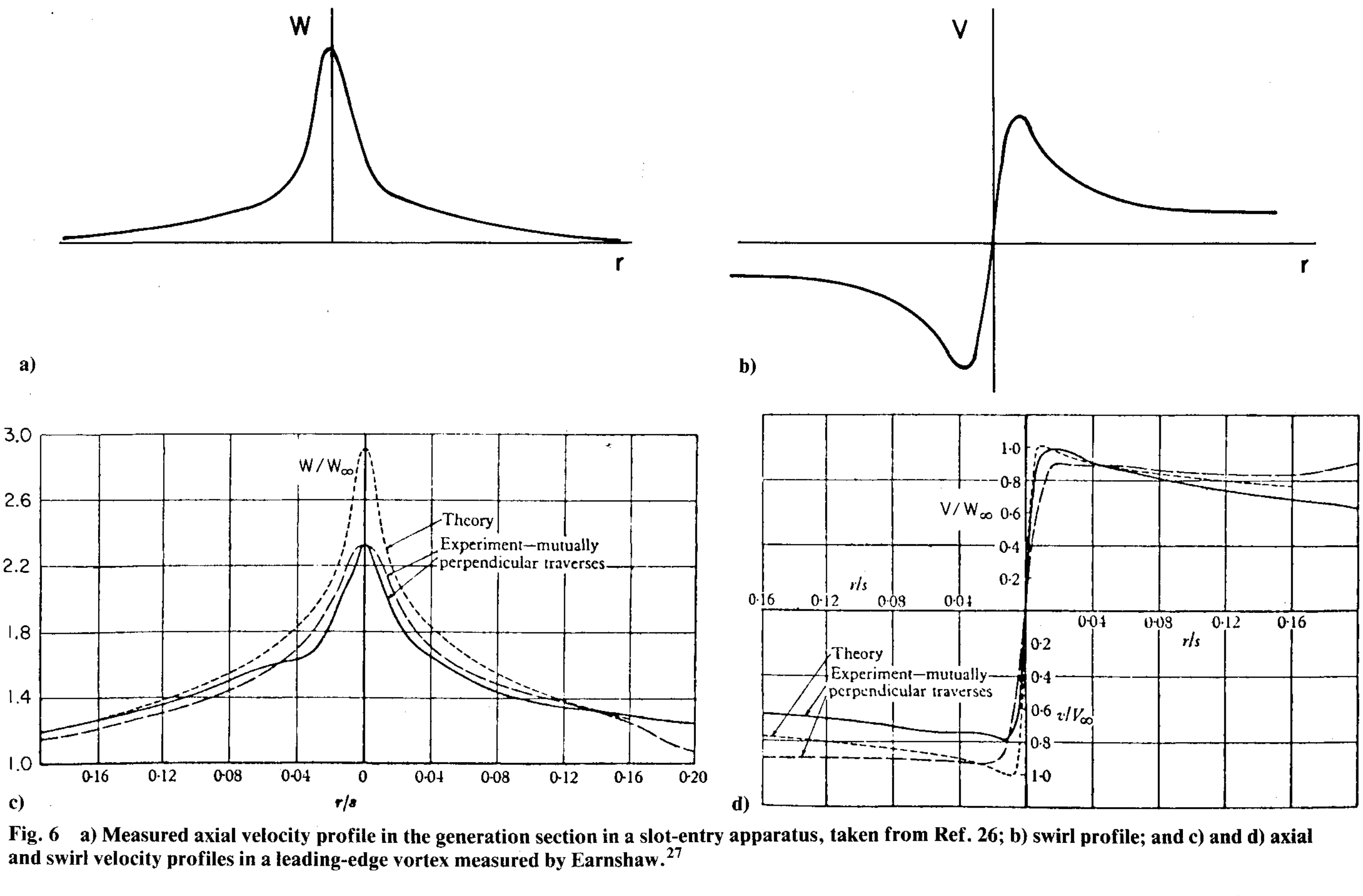

Fig. 6 a) Measured axial velocity profile in the generation section in a slot-entry apparatus, taken from Ref. 26; b) swirl profile; and c) and d) axial and swirl velocity profiles in a leading-edge vortex measured by Earnshaw.[27]

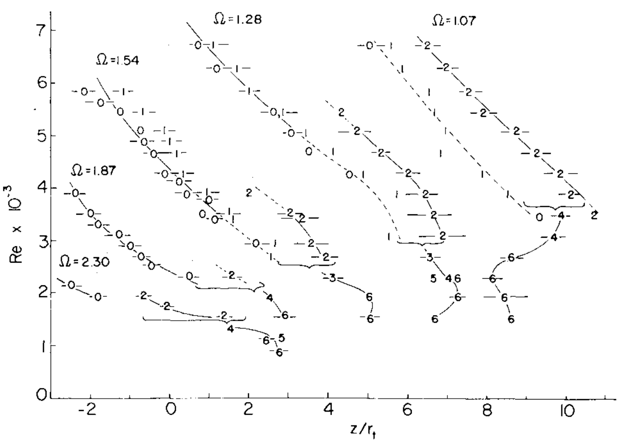

Fig. 7 Plot of the position (abscissa) of breakdown as a function of Reynolds number *Re* and swirl lead Ω, from Ref. 34. Positions marked 0 and 1 locate B-type breakdowns, while 2 locates S-breakdowns. Other designations (3-6) locate large amplitude disturbances that do not persist at high Reynolds number.

It is interesting to note that the sense of the helical dye filament of the leading edge S-breakdown is opposed to the direction of rotation of the upstream vortex, although the apparent direction of rotation of the spiral, as a geometric form, agrees with it.[5] Sarpkaya[30] and Faler and Leibovich[34] found the sense of the helix and its sense of rotation to agree with that in the approach flow in their experiments in vane-type devices. Escudier and Zehnder[26] found that the helix sense of S-breakdowns in their experiments agreed with the observations of Lambourne and Bryer. Since we have already argued that the flows generated by tangential jet entry devices are closer to leading-edge vortices than they are to flows in vane devices, the latter agreement is perhaps to be expected. Actually, it is not clear that the helix sense has much significance. The helix marks a thin filament of fluid particles surrounding the upstream vortex axis. As Lambourne and Bryer[5] have shown, individual fluid particles in the streakline as it expands radially downstream of the stagnation point have virtually zero swirl velocity, consistent with the need to conserve angular momentum. The formation of a spiral structure is consistent with a nonaxisymmetric displacement of the dye filament of the form $\exp[i(\theta+\omega t)]$ in the neighborhood of the stagnation point, with each dyed fluid particle thereafter moving on a surface of revolution in an essentially fixed meridional plane until it passes into regions of turbulent flow. The sense of the helix is determined by the sign of ω. The "rotation" of the geometric form can be accomplished without rotation of its constituent parts.

IV. Remarks on the Departure from Axial Symmetry of Vortex Breakdown

Approach flows which suffer breakdown can be made to have a high degree of axial symmetry, yet the flows in the breakdown zone and in the wake lose this symmetry. The loss of symmetry is associated with instability of the axisymmetric flow and presages transition to a turbulent wake. Thus, asymmetry shapes the properties of the wake, and, therefore, the global flow, but is the loss of symmetry intrinsic and necessary for the formation of a stagnation point and consequently an essential element in the mechanism of breakdown? The answer is not known, nor, of course, could it be in the absence of a definitive identification of the breakdown mechanism. Nevertheless, a correct working hypothesis could help in the construction of a theory which, it is to be hoped, will eventually be constructed as a comprehensive explanation of the phenomenon. As will be seen in the next section, virtually all of the theoretical attempts that have been made adopt the view that nonaxisymmetric effects are incidental, and that the dominant mechanism of breakdown is axisymmetric; Benjamin[7] has most explicitly articulated this idea. In this section, arguments will be given that might be marshalled to support either side of the question.

If we adopt a cylindrical (r,θ,z) coordinate system, we may represent any flow variable by a Fourier series in the azimuth θ in the form

$$\sum_{n=-\infty}^{\infty} C_n(r,z,t)e^{-in\theta} \tag{3}$$

Furthermore, if $C_n(r,z,t)$ are the Fourier coefficients of the axial velocity component, then $C_n(0,z,t)=0$ for all nonzero integers n. Thus, the only disturbance to the axial velocity on the axis of symmetry is that associated with the axisymmetric $(n=0)$ Fourier component. Since the variation with z of this Fourier mode of the axial velocity must be large to describe the arrest of the axial jet of the approach flow, it is apparent that any vortex breakdown, be it of S- or B-type, must have a large axially symmetric disturbance. Since the fluctuating velocities (indicative of the magnitude of the $n\neq 0$ Fourier components) are never more than a modest fraction of the axial jet in the approach flow, one, therefore, could assert that in this sense, the axisymmetric mode dominates the nonaxisymmetric modes. This is one argument in support of the position that nonaxisymmetric disturbances may play a secondary role in the mechanism of vortex breakdown.

A second argument is based upon the probable existence of exact, steady, axisymmetric solutions of the Navier-Stokes equations with reversed flow regimes that resemble vortex breakdown, and the known existence of physical flows with this characteristic. Laboratory experiments on flows in a closed cylindrical container, driven by a rotating lid, have been carried out by Vogel[39] and by Ronnenberg,[40] and reveal the existence of a closed, steady, axially symmetric region of reversed flow in a limited volume of fluid surrounding the axis of symmetry. Numerical solutions of the Navier-Stokes equations by Lugt and Haussling[41] successfully predict aspects of the observed flow. Of course, this is a flow dominated by wall friction; and the basic flow, as well as the flow reversal region singled out, is a recirculating one. Nevertheless, these numerical and experimental works show that "ideal" axisymmetric solutions containing features broadly resembling the ones we are interested in here do exist and can be experimentally realized. More closely related, but perhaps less firmly established, are numerical solutions of the steady axisymmetric Navier-Stokes equations by Kopecky and Torrance[42] and by Grabowski and Berger[43] intended to simulate representative vortex breakdown flows. Although questions can be raised[17] about the reliability of these solutions, they constitute substantive evidence of the existence of exact steady, axisymmetric vortex breakdown solutions.

One can begin, taking the other side of the question, by accepting both of the points made above, that 1) the axially-symmetric Fourier component of vortex breakdown flows may be significantly larger than the nonaxisymmetric ones, and that 2) the Navier-Stokes equations may admit steady, axially symmetric solutions with reversed flow regions. We will call the latter solutions of class A. The existence of such a class does not imply that the nearly axisymmetric flows observed in experiments, and admitted as point 1 here, need be close to any of the solutions of class A.

In fact, the sequence of successive transitions observed experimentally by Sarpkaya,[30] Faler and Leibovich,[34] Garg and Leibovich,[36] and Escudier and Zehnder[25] as the swirl level is increased suggests that both S- and B-forms of breakdown evolve from flows that are clearly not axially symmetric. The first large amplitude disturbance arises as a steady, nonaxisymmetric bifurcation of a steady, axisymmetric flow[34]; sequential increases of the swirl beyond this first critical level lead to a sequence of nonaxisymmetric flows with (or so it seems) an increasing axisymmetric aspect, although nonaxisymmetric features are never lost. This suggests that the members of class A are unstable solutions of the Navier-Stokes equations, and the flows realized in experiment may or may not be close to them. The momentum transport created by the symmetry-breaking modes arising from the instabilities can cause the axially symmetric part of the motion to differ drastically from solutions belonging to A, even though the nonaxisymmetric modes remain much smaller than the axisymmetric one. The Taylor-Couette flow between coaxial rotating cylinders provides an example of this in a simpler and more extensively studied setting. In this case, the symmetry-breaking perturbations are the Taylor cells which are themselves steady and axisymmetric at modest supercritical Taylor numbers; the symmetry is broken by the appearance of axial coordinate z rather than the azimuth angle. Even when the cellular motion is weak compared to the mean flow, the z-averaged velocity profiles differ significantly from the exact stationary cylindrical solution of the Navier-Stokes equations.

In summary, although the matter remains somewhat uncertain, it appears that axially symmetric breakdowns are admissible solutions of the Navier-Stokes equations, and these are physically realizable at sufficiently low Reynolds numbers in wall-dominated flows. On the other hand, maintenance of symmetry appears impossible at the high Reynolds numbers of aerodynamic interest, and the steady, axisymmetric solutions of the Navier-Stokes equations at high Reynolds numbers, assuming they exist, are unstable to nonaxisymmetric perturbations. Furthermore, the long-range effects of breakdown arise because of these instabilities; without them, the breakdown disturbance would be a local imperfection, annealed by the flow. A step that would help to clarify the question and which seems feasible, but only barely, is an investigation of the stability of numerically generated flows of class A to three-dimensional disturbances. This involves a challenging nonseparable eigenvalue problem. (Nguyen,[44] in a recent thesis, examined the viscous, spatial stability of solutions found by Grabowski and Berger,[43] with the velocity profiles at each station treated as though the flow were columnar. Such an effort, though interesting and valuable, is incapable of treating what is likely to be the most significant region of instability, the breakdown zone itself.)

V. Wave Propagation and the Criticality Criterion

Viewed on scales much larger than the vortex core diameter, vortex breakdown appears to be a discontinuous transition between two flows with very different characteristics. In this sense, it is reminiscent of a shock wave in gasdynamics or its analog in free surface flows, the hydraulic jump.

Waves may propagate in all three of these physical systems. In hydraulic jumps, in gasdynamic shock waves, and (for all adequately documented observations) in vortex breakdowns, transition separates an upstream flow (the supersonic or supercritical region) incapable of supporting upstream propagating wavelets, from its downstream successor (the subsonic or subcritical region), which admits both upstream and downstream propagating wavelets. Furthermore, the three systems share a common sensitivity to conditions imposed downstream. The occurrence and position of a shock in a nozzle is controlled by the back pressure, and the occurrence and location of a hydraulic jump in a channel of variable width is controlled by backwater conditions. While it is more difficult to control backwater conditions in vortex breakdown experiments, the effects of changing pressure gradients,[32] insertion of solid objects on the vortex axis, and deflection of trailing-edge flaps[5] on the occurrence and location of vortex breakdown indicate that this phenomenon has a similar sensitivity to downstream conditions.

Like a free surface flow, a vortex core admits infinitesimal dispersive waves. If the wavelengths of wavy disturbances are small compared to the length scale on which the core characteristics change, then the waves may be treated approximately as if they were propagating on a columnar vortex. Let the velocity field in such a columnar vortex be $[0, V(r), W(r)]$ in cylindrical (r,θ,z) coordinates. The radial velocity component of a wavelet has the form

$$u(r,\theta,z,t) = Au_0(r)\exp\{i[k(z-ct)-n\theta]\} + \mathrm{cc} \tag{4}$$

where $Au_0(r)$ is a complex amplitude (A being constant) describing the modal structure, cc stands for the complex conjugate of the preceding term, and $c=\omega/k$ is the phase speed of a wave of angular frequency ω in the z direction. Other flow disturbance variables in the wave have a similar form.

The interpretation of supercriticality or subcriticality for vortex breakdown flows is based upon the wave propagation characteristics of the mean flows; this is the same as the instantaneous flow upstream, but the instantaneous downstream flow is replaced by its time average, which is steady and axisymmetric. Furthermore, the classification, due to Benjamin,[7] is based upon propagation of axisymmetric ($n=0$) wavelets. Flows are supercritical according to Benjamin if the minimum phase speed c is positive [where it is assumed that the basic flow axial velocity $W(r)$ is everywhere positive], so that surfaces of constant phase cannot propagate upstream. It can be generally shown (see Leibovich[45]) that, if the flow is stable, there are two branches of the dispersion relation for each axisymmetric wave mode, one corresponding to phase speeds $c_-(k)$ less than the minimum axial velocity of the basic flow, and one having phase speeds $c_+(k)$ greater than the maximum axial velocity.

Benjamin[7] showed that the general effect of increasing the level of swirl is to reduce the minimum value of c_-, so that increasing swirl in a supercritical flow drives it toward critical and, if increased enough, the flow can be made subcritical. Since the medium is dispersive, the ability of perturbations to propagate upstream is determined by the group velocity $c_g = \partial\omega/\partial k$, not the phase velocity, and Benjamin's classification need not carry the interpretation given to it. Nevertheless, Leibovich[45] showed that Benjamin's criterion for supercriticality is always consistent with a definition based on group velocity; furthermore, since $c_g \to c$ as $k \to 0$, flows that are subcritical based upon a phase velocity criterion are also subcritical based upon the more fundamental group velocity criterion. Thus, Benjamin's classification based on phase velocities is coincident with one based on group velocities, at least if consideration is restricted to axisymmetric modes.

It is not generally known whether the group velocity of nonaxisymmetric modes is directed downstream in supercritical flows as defined in the preceding. If not, the concept of criticality is probably demolished. For the few special cases for which calculations have been made (Tsai and Widnall[46] for flow (2) with various profile constants determined by Garg[47]; and Leibovich and Ma[48] for flow (2) with $W_2 = 0$ and $\alpha = 1$), the classification remains intact.

Criticality Criterion of Onset

The wave propagation characteristics of a vortex core change with axial distance as the core evolves. This process is direct in a leading-edge vortex, since then the core vorticity increases along the axis. In the case of tubes of variable area, the vortex properties are changed by the geometrical constraints. Squire[6] proposed that breakdown first becomes possible when conditions in the vortex are critical. The model he used for his illustrative calculations was columnar and he did not discuss the evolution of core properties. Nevertheless, he must have had in mind the evolving vortex, and presumably he intended to suggest that breakdown would occur somewhere near the spot where critical conditions arise. Thus, the presence of a downstream source of disturbance, such as a trailing edge of a wing, or a sudden change in tube area, would be communicated by waves propagating upstream through the subcritical section of the vortex waveguide, but would be unable to proceed further than the station at which critical conditions obtain.

Benjamin[7] criticized this conception, pointing out that the group velocity at criticality is positive in the downstream direction, so that in a columnar flow, disturbances cannot spread upstream from a disturbance source. Had Squire described the processes in an evolving vortex core, along the lines above, Benjamin's objection would not have seemed compelling.

As a first approximation, the location at which critical conditions occur in a vortex can be taken as an estimate of breakdown location. This may be refined by recognizing that the speed of a nonlinear wave increases linearly with its amplitude, so that a nonlinear wave can penetrate into the supercritical region of the vortex. Thus, a more general wave theory will locate a breakdown somewhat upstream of the critical station. Further discussion of this point is deferred to the latter part of this section.

The critical condition for flows with given velocity profiles can be computed with little effort from a second-order ordinary differential equation. As an example, the regions of supercritical and subcritical flow for a vortex model (2) have been computed (using boundary conditions appropriate to unbounded domains, see Leibovich[11]). The velocity profiles of the model may be given in rescaled form by

$$V(r) = q\,|\delta|\,r^{-1}\,[1-\exp(-r^2)] \qquad (5a)$$

$$W(r) = 1+\delta\,\exp(-r^2) \qquad (5b)$$

where q is a positive constant, $\delta>0$ in an approach flow (jet), and $\delta<0$ in a wake flow. As seen in Eq. (5b), the axial velocity tends to a constant as $r\to\infty$, and this speed has been used to normalize the velocities, while distances are normalized by the radial distance at which the axial vorticity e-folds once, a measure of the core size that turns out to be close to the radius of maximum swirl. The parameter q in Eq.(5a) was introduced by Lessen et al.[49] in their treatment of the stability of this flow, and its value completely determines the inviscid stability characteristics. Consequently, q is retained in the description and Eq. (5) will be called the "Q-vortex" henceforth.

The results of the calculation are shown in Fig. 8. Also shown there are regions of instability according to the theories summarized in Sec. VI, as are the approach-wake transitions reported by Garg and Leibovich[36] as deduced from their Table 1. For convenience, the reprocessed data are compiled in Table 1, where the symbol A identifies approach flow and W the wake flow. The Reynolds number is given first with respect to the tube radius and, in parentheses, with respect to the core diameter based on maximum swirl (as in Leibovich,[17] Table 1). The core expansion ratio[17] is also shown: the jump in q from higher values upstream to lower ones downstream is consistent with the mixing occurring in the wake, which causes an increase in core diameter and consequent reduction in peak core vorticity (hence, a reduction in q).

Notice that all approach flows are supercritical. The "distance" from criticality is best measured by the Froude-like number N, introduced by Benjamin,[7]

$$N = (c_+ + c_-)/(c_+ - c_-) \qquad (6)$$

where c_+ and c_- are the greater and lesser values of the phase speed for waves of extreme length. For prescribed values of δ and q, N may be found from the data in Fig. 8. To find (c_+, c_-) for a noncritical flow with given (δ, q), relabel the abscissa, letting it be ℓ; and let the positive branch of the continuous curve separating sub- and supercritical regions be ℓ_+, and the negative branch be ℓ_-. Given q, two values $\ell_\pm(q)$ are defined, and $c_+ = \max(1-\delta\ell_+, 1-\delta\ell_-)$, $c_- = \min(1-\delta\ell_+, 1-\delta\ell_-)$. In wake flows, $-1<\delta\le 0$, and $\ell_- < -1$. It is understood that flows with axial velocity reversals are excluded from the discussion.

Conjugate Flow States

For some applications, including aerodynamic ones, the details of the transition region are not as important as the position of breakdown and the general properties of the flow following it. A method of determining the wake properties from the approach flow, like shock jump conditions, may suffice.

Benjamin's[7] theory of conjugate flows does not provide jump conditions, but it does aim at relating the downstream flow to the approach flow without considering the structure of the transition. The theory is cast in terms of steady, inviscid, and axisymmetric motion, which, as is well known (cf. Ref. 7), is described by a single equation for the Stokes streamfunction ψ. In cylindrical coordinates, the equation is

$$\psi_{rr} - r^{-1}\psi_r + \psi_{zz} = r^2 H'(\psi) - KK'(\psi) \qquad (7)$$

where subscripts indicate partial derivatives. H is the total head and (except for a factor of 2π) K is the circulation; both are functions of ψ alone,

$$H = p/\rho + \tfrac{1}{2}(u^2+v^2+w^2), \qquad K = rv \qquad (8)$$

where the (r,θ,z) velocity components are (u,v,w). The functional forms of H' and K may be determined if all streamlines intersect a plane $z=$ const upon which the pressure and velocity are known. These functions typically are highly nonlinear (and not Galilean invariant). For example, the vortex Eq.(2) with $W_2=0$ has

$$H'(\psi) = \alpha\lambda^2 e^{-\beta\psi}\,[1-e^{-\beta\psi}]$$

$$F = \lambda[1-e^{-\beta\psi}], \qquad \beta = 2\alpha/W_1 \qquad (9)$$

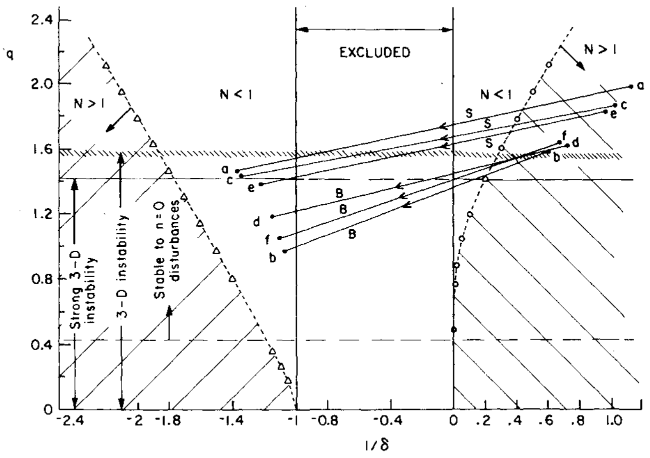

Fig. 8 Parameter space for the Q-vortex, showing the regions of subcritical and supercritical flow, regions of stability to axisymmetric disturbances, and regions of instability to three-dimensional disturbances. Strong instability to three-dimensional disturbances is found for $q<\sqrt{2}$, and this region is shown. Also shown in the plot are the approach-wake transition for the three sets of data given in Table 1.

If $W_1 = 0$, in the same model,

$$H'(\psi) = 4\alpha^2\psi\{1 - \alpha q^2/\ln[1 - 2\alpha\psi/W_2]\} - 2\alpha W_2$$
$$K = 2\alpha q\psi, \qquad q = \lambda/W_2 \tag{10}$$

while Hall's[21] conical vortex model has

$$H'(\psi) = C\psi^{-1} \qquad K(\psi) = \lambda\psi^{1/2} \tag{11}$$

where C and λ are constants.

Benjamin[7] considers columnar solutions $\psi = \psi(r)$ in the cylinder $0 < r < a$. Assuming that a supercritical flow $[0, V(r), W(r)]$ is prescribed, the corresponding functions H and K can be found, and the prescribed supercritical flow is obviously a solution of Eq. (7). Typically, other solutions to Eq. (7) will exist, however, with the same functional forms for H and K; if so, then Benjamin shows that they are necessarily subcritical and have larger values of the momentum flux

$$S = 2\pi\int_0^a (p + \rho W^2) r\mathrm{d}r$$

which Benjamin calls the "flow force." These properties were place on a rigorous mathematical footing by Fraenkel.[50] The defining supercritical flow and a second solution (assuming it exists) with S closest to the value of S of the defining flow are called a conjugate flow pair.

The conjugate flows do not represent an analog of a shock jump condition, however. Since momentum must be conserved, two flows in the same cylindrical space can be connected only if they have the same value of S (neglecting wall friction). This is possible only if energy is dissipated or radiated away by the (unsteady) formation of waves in a transition region that is outside the scope of the analysis. Benjamin[7] shows that the formation of waves leads to a reduction in S and his conception of idealized vortex breakdown is the transition between an upstream supercritical flow and a downstream flow consisting of the subcritical conjugate and a superposed wavetrain.

As in the theory of normal shocks, the level of supercriticality, measured by the Froude number N, at which a breakdown occurs cannot be specified, but Benjamin argues that slightly supercritical conditions are most dangerous for the occurrence of "spontaneous" breakdown.

Hall[9] criticized Benjamin's theory on the grounds that it does not offer an advance of Squire's[6] simpler idea in providing a prediction of the occurrence and location of breakdown; and that the concept of reduction of S by waves is supported only by weakly nonlinear theory, whereas the perturbations occurring in breakdown are necessarily large. The most serious weakness of Benjamin's[7] theory in the view of this author is that there is no clear way to relate it to experiments which, at high Reynolds numbers, always have unsteady, nonaxisymmetric wakes. The only conceivable way to relate the theory to experiment is to regard the streamfunction appearing in the theory as the streamfunction for the time-averaged flow in the wake, which is axially symmetric and steady. If this is done, however, there is no need for the total pressure $H(\psi)$ in the wake to have the same functional form as it has in the supercritical defining flow upstream. The correspondence between the functions for H in the conjugate states can be destroyed simply by unsteadiness, and does not require energy dissipation. Thus, to accept Benjamin's theory, which has provided such a wealth of useful concepts, requires an act of faith for which no scientific basis has yet been provided.

Nonlinear Wave Models

The linear long wave model of Squire[6] leads to the critical condition. If the basic vortex flow forms a waveguide with slowly changing characteristics, then, as outlined above, disturbances created downstream can penetrate upstream, the long waves coming to rest near the section of the vortex where critical conditions are reached. This picture of breakdown as a wave provides a simple conceptual framework that explains qualitative features concerning the location of breakdown and its movement in response to the variation of experimental control parameters.

Without accounting for finite wavelength, however, the region of wavy disturbances can have no beginning in a waveguide of fixed characteristics, the case explicitly treated by Squire; while if the waveguide varies, the long waves have a singularity at the critical station entirely analogous to that arising in linearized gas dynamics at the sonic condition.

The removal of these difficulties requires the inclusion of finite amplitude and finite wavelength effects and provides a wave model that is free of these inconsistencies. Furthermore, the enlarged wave theory shows that the position of breakdown, regarded as the location at which the wave is arrested, lies in the supercritical region of the waveguide and allows some structural information in the breakdown zone to be inferred.

Nonlinear wave theory is developed and applied to vortex breakdown in Refs. 10-13. The first two references deal with columnar waveguides, while the second two deal with slowly varying vortex waveguides and provide a "transcritical" theory for the problem related, rather distantly, to transonic theory.

Benjamin[10] first showed that axially symmetric solitary wave solutions are possible in a columnar vortex, that they are the only disturbances of permanent form that can arise in the absence of energy loss, and that they are stationary in a supercritical stream. Benjamin's paper is mainly concerned with infinite wavetrains of finite amplitude; the only example of a solitary wave presented leads to an acceleration of flow along the vortex axis rather than a deceleration as required for breakdown. Leibovich,[11] in a further development of Benjamin's work, showed that axisymmetric waves of small but finite amplitude propagating on a columnar vortex have a Stokes streamfunction of the form

$$\psi = \Psi(r) - \epsilon\phi_0(r)A(z,t) \tag{12}$$

where $\Psi(r)$ is the streamfunction of the unperturbed flow, ϵ a small amplitude parameter, and $\phi_0(r)$ the eigenfunction of a second-order ordinary differential equation. The wave amplitude is governed by the Korteweg-deVries equation

$$A_t + c_0 A_z = \epsilon\{\alpha A A_z + \beta A_{zzz}\} \tag{13}$$

where c_0, the linear long wave speed, and the constants α and β are all functionals of the unperturbed flowfield that may be computed. For radially unbounded flow with vorticity vanishing at distances large compared to the core radius, A is determined instead by an integrodifferential equation of the same form as Eq. (13), but with the term proportional to $\partial^3 A/\partial z^3$ replaced by

$$\kappa\frac{\partial^3}{\partial z^3}\int_{-\infty}^{\infty}\frac{A(\xi,t)\mathrm{d}\xi}{[(z-\xi)^2 + \ell^2]^{1/2}} \tag{14}$$

where ℓ is the ratio of the core radius to the wavelength and κ is a constant (depending upon ℓ) that is fixed by the theory.

The Korteweg-deVries equation has solutions describing waves of permanent form, the "solitons," which evolve from initial conditions of rather general type; they are the solitary waves known since the nineteenth century. The soliton with unity as maximum value of the amplitude A propagates, in the notation used in Eq. (13), at a speed $c_0 - \epsilon\alpha/3$ if $\epsilon\alpha$ is positive, as it turns out to be in examples of realistic basic vortex flows. The dependence of wave speed on wave amplitude is again analogous to gasdynamics, but here the medium is dispersive and waves of permanent form can develop without dissipation. From this dependence, one concludes[10,11] that the soliton is stationary if the flow upstream of it is supercritical. If the amplitude parameter ϵ is allowed to take any value, instead of the small ones for which the theory has asymptotic validity, then the soliton can create a stagnation point on the axis and axial flow reversal.[11]

Being close to critical has no effect on waves in a cylindrical tube, but has a large effect on waves in a tube of variable area. Leibovich and Randall[12] showed that a different procedure is needed to find the structure of weakly nonlinear waves in nearly critical, or "transcritical" flow; their theory was applied to breakdown by Randall and Leibovich[13] when the deviation, $g(z)$, of the tube area from its maximum value is small and changes slowly with z, so $(dg/dz)/g$ is small compared to the wavelength of the disturbance. The equation describing waves on a nearly critical flow in such a slowly varying waveguide is (in notation differing from the original)

$$A_t = \epsilon\{\alpha A A_z + \beta A_{zzz} - \sigma(gA)_z - \nu A\} \tag{15}$$

Here α, β, σ are, again, computed constants, and ν is a computed constant times an inverse Reynolds number based upon wavelength. The consequences for breakdown ensuing from the adoption of Eq. (15) have been explored[13] and the associated Stokes streamfunction and circulation

$$\psi = \Psi(r) + \phi_0(r)[g(z) - \epsilon A(z,t)]$$

$$rv = K(\Psi) + \gamma_0[g(z) - \epsilon A(z,t)]$$

(where γ_0 is given in terms of ϕ_0), as a model for the flow without restriction to small ϵ. The presence of dissipation, represented here by ν, turns out to be important. Molecular diffusion, as assumed in Randall and Leibovich,[13] is probably negligible as a source of dissipation in vortex breakdown at high Reynolds numbers, but the author believes that another form of "apparent dissipation," to be explained in Sec. VII, is significant and that ν should be based upon it.

Equation (15) has solutions resembling solitary waves. Assuming the tube diverges in the downstream direction, the area changes cause waves to amplify as they propagate upstream, and decay as they move downstream. The possible ultimate equilibrium position of a finite amplitude wave is closely estimated by the roots of the equation

$$\frac{\sigma dg}{dz} + 2\nu = 0 \tag{16}$$

In the problem considered in Ref. 13 (which was thought to be a reasonable model of Sarpkaya's[30] experiments), $\sigma > 0$. If ν is to represent dissipation, then $\nu > 0$, and the solutions must lie in a portion of the waveguide where $dg/dz < 0$, or where the tube is diverging. For problems of flow in an initially straight tube that flares out and then becomes straight again, there are two equilibrium points z_1 and $z_2 > z_1$. Only the upstream point is stable, however, in the sense that waves passing it in either direction experience a restoring tendency owing to the amplitude-speed behavior of the waves.

If the wave is to be stationary at z_1, it must have a specific amplitude. Without loss of generality, the maximum value of A can be set at unity, and the condition that the wave be stationary at z_1 then fixes the amplitude parameter at

$$\epsilon = 3g(z_1) \tag{17}$$

For the conditions obtained in experiments containing vortex breakdown, the computed value of ϵ is not small and the resulting flow calculated contains a stagnation point. For the basic flow model selected as representative of Sarpkaya's experimental data, a streamline pattern containing a stagnation point and a recirculation bubble was found. The result was interesting because there are no adjustable constants in the theory; the fact that a reversed flow region exists is apparently the only possibility if the disturbance is to remain stationary! Furthermore, the core radius of the basic vortex and the radius of the bubble of reversed flow are approximately the same as observed in experiment.

Much has been made, mostly in Refs. 10 and 17, of the fact that the sense of the swirl velocity reverses as one crosses a dividing streamline of a recirculation zone embedded in an inviscid swirling flow if the functional form of the circulation $K(\psi)$ [in the notation of Eq. (6)] is analytically continued from the outside of the closed streamline region where it is defined to the inside. This reversal, being impossible without an applied torque, was considered a serious weakness of the interior solutions found for the regions of reversed flow. It is now believed that this qualitative inconsistency is easily removed. It has always been known (see Refs. 17 and 51 for discussions in the context of vortex breakdown) that a different functional form for $K(\psi)$ may be chosen for the interior. This cannot be done arbitrarily if the shape of the dividing streamline is to be maintained, because the interior and exterior pressures must balance on all points of the streamline, and an alteration of $K(\psi)$ alters the pressure distribution. Having found a dividing streamline using a single functional form for $K(\psi)$, however, one can maintain its shape by the replacement of $K(\psi)$ by $-K(\psi)$ in the interior; all dynamical balances are thereby unchanged and no swirl reversal then occurs.

VI. Instability of Inviscid Columnar Vortices

It is fortunate that great stretches of vortex flows, even those in which breakdown occurs, can be approximated as columnar, since the columnar vortex is analogous to the parallel model of two-dimensional basic flows. It is also fortunate that experimentally observed velocity profiles of vortices undergoing breakdown, at least in some carefully studied examples, can be fitted reasonably well by the analytically simple family of profiles given by Eq. (2) (the Q-vortex), and that some results are known about the stability of this family. The stability analysis of the Q-vortex is intricate and delicate and a complete characterization is not yet available. Nevertheless, what is known about its stability, and, then, general criteria for stability and instability of columnar vortices will be reviewed. Attention will be restricted throughout to inviscid motions. Finally, the application of the results to the analysis of experimental data in vortices suffering breakdown will be discussed.

Suppose the basic flow to be analyzed is described in cylindrical (r,θ,z) coordinates by the velocity vector $v = [(0, V(r), W(r)]$. If the flow is inviscid, the functional forms of V and W are arbitrary. The problem governing the linear inviscid temporal stability of columnar vortices to three-dimensional disturbances of normal mode form $\exp[i(kz - n\theta - \omega t)]$ is the Howard-Gupta[52] eigenvalue problem

$$D[SD_* u] - \{1 + a(r)/\gamma + b(r)/\gamma^2\}u = 0$$

$$u(R_1) = u(R_2) = 0 \tag{18}$$

where R_1 and R_2 are the boundaries, $0 \le R_1 < R_2 < \infty$, $D(\ ) \equiv d(\ )/dr$, $D_*(\ ) \equiv D(\ ) + (\ )/r$, and $a(r)$, $b(r)$, and $S(r)$ are prescribed functions of the basic flow, and of the assigned real wavenumbers k and n. The function γ is the (negative) Doppler-shifted frequency, and is given by

$$\gamma = k \cdot v - \omega \tag{19a}$$

$$k = (0, -n/r, k) \tag{19b}$$

where k is the wavenumber vector of the assumed normal mode, and the frequency ω is to be found as the eigenvalue. The function $S(r)=1/(k\cdot k)$ is just a geometrical quantity. The functions $a(r)$ and $b(r)$ are given by

$$a(r)\equiv r\mathrm{D}[S(r^{-1}\mathrm{D}\gamma-nr^{-3}V)]$$

$$b(r)\equiv -2kr^{-2}VS[kr\mathrm{D}_*V+n\mathrm{D}W] \tag{20}$$

The latter quantity plays an important role in the work to be described in this paper. It is of interest to note here that $b(r)$ has a simple kinematic interpretation. Let ζ be the vorticity vector of the basic flow and let $\Omega=(V/r)e_z$ be the angular velocity vector of a ring of fluid particles of radius r in the basic flow, where e_z is a unit vector in the direction of z increasing, and let $e_w=k/(k\cdot k)^{1/2}$ be a unit vector normal to the wavefronts of the assumed normal mode disturbance. Then,

$$b(r)=-2(e_w\cdot\Omega)(e_w\cdot\zeta) \tag{21}$$

Studies of the Q-Vortex Model

Lessen et al.[49] investigated the stability of the flow with velocity vector v, given in cylindrical coordinates (r,θ,z) by

$$V=qr^{-1}[1-\exp(-r^2)];\qquad W=\exp(-r^2) \tag{22}$$

where r ranges from zero to infinity. There is no loss in generality if q is taken to be positive, and this will be assumed. This form serves (by a Galilean shift) as a model of flows with either jet-like or wake-like axial velocity components of Eq. (2) or its equivalent, Eq. (5). The Galilean shift does not affect stability questions, nor, therefore, does the jet-like or wake-like character of the axial velocity profile; stability or instability is characterized only by the parameter q. Similarly, the stability of a columnar vortex with another set of profiles seems to depend upon the minimum value over r of the quantity $q=-r\mathrm{D}_*V/\mathrm{D}W$ which is constant for the Q-vortex.

According to the results of numerical studies by Lessen et al.[49] (supplemented by additional calculations by Duck and Foster,[53] and by Leibovich and Stewartson[54]), the Q-vortex is unstable to three-dimensional disturbances with normal modes of the form $\exp[i(kz-n\theta-\omega t)]$ provided $q<q_I$. The precise value of q_I (indeed, the existence of this upper bound) is unknown, but it is known to exceed 1.58. Modes with positive azimuthal wavenumbers n are found to be the most dangerous.

A plot of growth rate $[\omega_i=\mathrm{imag}(\omega)]$ against axial wavenumber (k) found by Lessen et al.[49] for $q=0.8$ is shown in Fig. 9. Notice that the maximum growth rate increases with n, at least up to the largest value (6) of n considered. This makes one wonder if ω_i continues to increase with n indefinitely, or whether there is a most unstable wave at a particular value of n. If the former case holds, does ω_i approach a finite limit as $k\to\infty$, or does it increase without limit? These questions were considered by Leibovich and Stewartson.[54] They showed that, on energy stability grounds, ω_i cannot exceed the maximum value of the rate of strain, s, of the basic flow, where

$$s=[(r\mathrm{D}\Omega)^2+(\mathrm{D}W)^2]^{1/2}$$

$\Omega\equiv V/r$ is the angular velocity of the basic flow. Furthermore, detailed considerations of the asymptotic behavior obtained as $n\to\infty$ revealed that ω_i approached a value ω_{∞_i} from below in the limit $n\to\infty$. Although originally derived for the Q-vortex, the result generalizes and holds for arbitrary profiles $V(r)$ and $W(r)$ and does not require that the flow be in a radially infinite region. The generalized value is

$$\omega_{\infty_i}=\max[-V\mathrm{D}\Omega(\mathrm{D}W/s)^2[2+J(r)]]^{1/2}$$

$$J(r)=2(\mathrm{D}\Omega)\mathrm{D}(rV)/(\mathrm{D}W)^2 \tag{23}$$

The function $J(r)$ is a local Richardson number, based upon the geometric mean of the angular velocity gradient $\mathrm{D}\Omega$ and the circulation gradient $\mathrm{D}(rV)$, which should be compared to the Richardson number arising in the stability criterion of Howard and Gupta[52] (to be discussed in the next section), in which $\mathrm{D}\Omega$ is replaced by Ω/r.

General Stability Criteria for Columnar Vortices and Further Results for the Q-Vortex

If the flow is inviscid, V and W can have arbitrarily prescribed functional forms. The special cases having $W(r)=0$ will be referred to as "pure" vortices. Stability results of a general nature are as follows:

1) From Refs. 55-57: A pure inviscid vortex is stable to infinitesimal axisymmetric perturbations if[55,56] and only if[57]

$$\Phi\equiv r^{-3}\mathrm{D}(K^2)>0 \tag{24}$$

where $K(r)\equiv rV(r)$ as in previous sections of this paper.

Note that Synge's contribution,[57] which is a sufficient condition for instability, is more definite than Rayleigh's,[55,56] which is a sufficient condition for stability, since one cannot guarantee stability to nonaxisymmetric perturbations from Rayleigh's work.

2) From Refs. 55 and 58: A necessary condition for the instability of a vortex, to nonaxisymmetric two-dimensional disturbances is that the vorticity, $\zeta=r^{-1}\mathrm{D}\Gamma$, have an extremum in the interior of the fluid (with the axis $r=0$ considered a boundary point, if it is not excluded from the flow domain). This is the inflection point criterion applied to swirling flows. Thus, a vortex with monotonic vorticity is stable to two-dimensional disturbances.

This condition is strengthened by a refinement of Fjortoft's.[58] A necessary condition for instability of a pure vortex to two-dimensional disturbances is that

$$[\Omega(r)-\Omega(r_I)](\mathrm{D}\Gamma)<0 \tag{25}$$

in some region in the flow, where r_I is a point at which $\mathrm{D}\zeta=0$.

As in parallel flows, this argument can be taken a step further (although we are not aware of its having been done in the literature) to virtually ensure instability if the angular velocity profile is monotonic, the vorticity ζ has a single

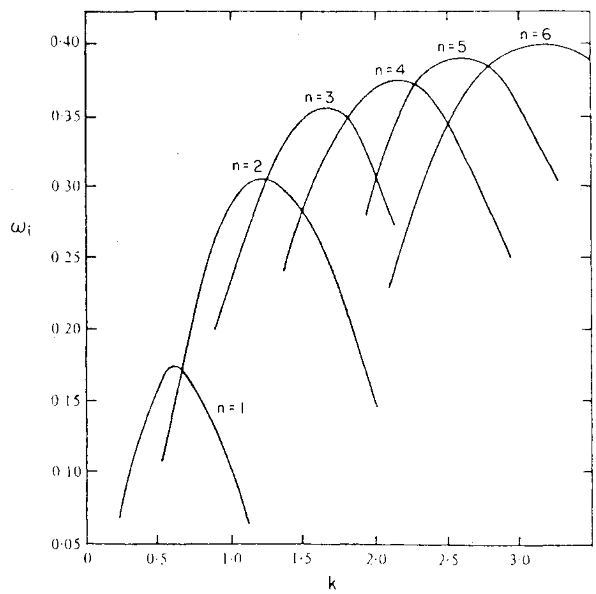

Fig. 9 Growth rates as a function of wavenumber k for Q-vortex for $q=0.8$ and for azimuthal wavenumbers 1-6, from Ref. 49.

extremum in the flow at $r=r_l$, and

$$[\Omega(r)-\Omega(r_l)](D\Gamma)<0 \quad (26)$$

throughout the flow. Then, the equality holds only at $r=r_l$ and, by a well-known argument due to Friedrichs, a regular neutral mode exists with $\omega=-n\Omega(r_l)$, where n is the azimuthal wavenumber of normal modes behaving like $\exp[-i(n\theta+\omega t)]$. An equally well-known calculation of Tollmien's allows one to compute contiguous unstable modes, assuming they exist. Since the likelihood of a regular neutral mode being isolated is small, Eq. (26) provides a reasonable expectation that a flow of this kind is, in fact, unstable.

3) From Ref. 52: In the presence of an axial velocity component, Howard and Gupta were able to generalize Eq. (24) to one of Richardson number type. They show that if

$$J\equiv\Phi/(DW)^2>1/4 \quad (27)$$

then the flow is stable to axially symmetric disturbances. For the Q-vortex, this guarantees stability to infinitesimal $n=0$ disturbances if $q>0.403$.

They also derived a semicircle theorem for axisymmetric perturbations which guarantees that unstable modes, if they exist, have critical layers. Barston[59] has produced a more flexible method of deriving semicircle theorems which is applicable to swirling flows with nonaxisymmetric perturbations. No general conclusions concerning critical layers seem to have emerged yet from that work, however.

4) From Ref. 60: Swirling flow in a pipe or annulus of radius a is unstable if

$$V(a)\neq 0, \qquad DW(a)\neq 0 \quad (28)$$

This result is obtained by generalizing the asymptotic analysis (as $n\to\infty$) of the problem of a rigidly rotating Poiseulle flow in a pipe.

5) From Ref 54: A sufficient condition for the instability of a columnar vortex is

$$VD\Omega[D\Omega D\Gamma+(DW)^2]<0 \quad (29)$$

Condition 5 can be related to Rayleigh's mechanism of centrifugal instability, but here an axial velocity is permitted, and the perturbations are not restricted to axially symmetric ones. This connection has already been made by Ludwieg,[14] who derived it for flow in a narrow annular gap by ingenious physical reasoning. Ludwieg's derivation of Eq. (29) did not permit its status (necessary or sufficient for stability) to be determined, and an "improved" criterion derived mathematically by Ludwieg[15] for inviscid flows in a narrow annular gap is not applicable to general flows. We will now show that Eq. (29), which follows a consequence of Eq. (23), analytically derived by Leibovich and Stewartson,[54] is an indication of a centrifugal instability. First, we note that at each value of the radius, the basic flow has one principal axis of its rate-of-strain tensor which corresponds to zero rate of strain. This axis is tangent to the cylindrical surface $r=$ const and its direction is indicated by the vector $\boldsymbol{\nu}$ in Fig. 2. Positive values of the growth rate Eq. (23) are possible if, in the normal mode analysis, there is a radial position $r=r_0$ in the basic vortex for which the coefficient $b(r_0)$, as given by Eq. (20), is positive, and for which the wavenumber vector $\boldsymbol{k}$ of the disturbance as given by Eq. (19b) is parallel to the principal axis corresponding to zero rate of strain of the basic flow. In a spiral coordinate system with one axis aligned with the direction of wave propagation, the flowfield appears to be locally two dimensional at $r=r_0$. Furthermore, if the basic flow vorticity and angular velocity are projected onto the direction of wave propagation, the function $b(r)$ is the negative of Rayleigh's parameter. If the flow is locally a pure vortex and $b(r)>0$, then the Rayleigh/Singh stability criterion is violated, and the flow is centrifugally unstable. This is the local picture at $r=r_0$ in the curvilinear coordinate system described; the existence of such an r_0 with $b(r_0)>0$ is guaranteed when Eq. (29) is satisfied.

Criteria 1-5, as far as known, are the only general stability results available for inviscid, incompressible vortex flows with continuous velocity profiles. It is notable that most of them are restricted to either pure vortices or to axisymmetric disturbances. Detailed numerical studies of special cases in both inviscid and viscous flows indicate that axial velocity shears are destabilizing[49,53,61-66] and that the most dangerous modes are not axially symmetric. The only definitive general results known to be applicable to these flows are 4 and 5.

On the other hand, these results, both sufficient conditions for instability, are known from special cases not to be necessary for instability. For example, for the Q-vortex, criterion Eq. (29) is satisfied for $q<\sqrt{2}$ and is not satisfied for larger values of q, but it is known from numerical computations that the flow is unstable, at least for the smaller values of n, for values of q larger than $\sqrt{2}$. Furthermore, marginal instability of the inviscid flow is likely to be determined by the $n=1$ mode, and will occur at $q=q_l$; q_l, assuming that a finite value exists, is known to exceed 1.58.

When Eq. (29) is satisfied, the analysis of Leibovich and Stewartson[54] shows that inviscid modes with arbitrarily large azimuthal wavenumbers are excited. The numerical evidence for the vortex model Eq. (22) further shows that if an azimuthal mode n_l is excited, then all modes with $0<n<n_l$ are also excited. In such cases then, the satisfaction of criterion Eq. (29) implies that in the absence of viscosity an infinite number of modes simultaneously have a finite growth rate. This suggests that for large Reynolds numbers, Eq. (29) is a criterion for massive instability and transition to turbulence.

Application of Stability Results to Vortex Breakdown Flows

Garg and Leibovich[36] report experiments upstream and downstream of vortex breakdowns of both the B- and the S-forms referred to earlier. In addition to mean flow measurements, they also report on an extensive set of measurements of power spectra that reveal strong low-frequency periodic oscillations in the wakes of both breakdown forms; similar oscillations had been reported earlier by Faler and Leibovich[17] to occur in the interior of the recirculation zone of a B-form of breakdown. Because the mean velocity profiles in both the approach and wake flows can be fitted with good accuracy to the Q-vortex profiles, they were able to apply the results of Lessen et al.[49] to show that the oscillation frequency in the wake agreed with the frequency of the $n=1$ mode. The upstream measurement stations all were sufficiently far from the breakdown zone to ensure that the assumption of columnar flow was a good one. Traverses at several downstream stations were taken, and the mean data at the stations sufficiently far downstream were thought to be columnar to an acceptable approximation. Furthermore, the values of q in the wakes in all experiments performed were less than 1.58, therefore, less than the critical value for instability to nonaxisymmetric disturbances. Improvements in the accuracy of the comparison between the frequencies of this experimental data base and the values from stability theory have been described by Tsai and Widnall.[46] Upstream of B-forms of breakdown, q was about 1.6; since we know that the critical value of q exceeds 1.58, the flows upstream may be unstable according to stability theory; in any event they are at best marginally stable. (It should be noted at this point that, although Leibovich and Stewartson found instabilities at $q=1.58$, the corresponding growth rates were very small.) The values of q in the wake of B-breakdowns were all less than $\sqrt{2}$, indicating strong instability, according to Leibovich and Stewartson.[54]

Corresponding measurements for S-type breakdowns revealed significantly larger values of q upstream, indicating an approach flow with a higher degree of stability, while the mean flows in the wake were decidedly larger than values in the wake of B-breakdown, but grouped near $\sqrt{2}$, and, therefore, definitely unstable according to stability theory. One concludes that the approach and wake flows of B-breakdowns are less stable than those of S-breakdown. The core expansion is larger in B-breakdown than in S-breakdown, and since this is likely to be the consequence of turbulent mixing, this provides additional corroboration of the stability conclusions.

The jumps in q across the breakdowns in the Garg-Leibovich data base are summarized in Fig. 8. It may be useful to emphasize that a larger q does not mean a larger swirl level; q is directly proportional to the peak swirl speed and inversely proportional to the axial velocity excess or deficit. The axial velocity excess or deficit is itself a strong function of swirl (see Faler and Leibovich, p. 1387, for a brief discussion).

Escudier et al.,[26] analyzed their experimental measurements in terms of its instability and criticality. The experiments were conducted in an apparatus employing a slot-entry tangential-jet vortex generator, followed by a sudden contraction, their "exit tube," in which breakdown occurred. They concluded that the flow was always both critical and unstable at the entrance to the exit tube, hence, upstream of breakdown. These interesting conclusions are apparently based upon two distinct approaches. First, there was visual evidence of instability at the entrance to the exit tube. Second, q (assuming the flow to be a Q-vortex) and the state of criticality (apparently assuming the flow to be a Rankine vortex with uniform axial speed) were estimated at each axial station at which data was available. These indicated a rapid fall of q as the contraction was approached, with values falling below 1.5 (the value they used as a criterion for loss of stability) just upstream of the entrance to the exit tube. The value of q was estimated by the rule (appropriate for the Q-vortex) $q = 1.56\ V_{\max}/W_2$, where $V_{\max}$ is the peak swirl speed, easily determined from the data.

The conclusions drawn by Escudier et al.,[26] in the author's view, are premature. It is not appropriate to apply a theory for columnar vortices to a region of flow with large axial gradients, as they have done, to arrive at the conclusion that q falls as the contraction is approached. Even if the flow were accurately columnar, they found it difficult to fit the measured mean axial profile (which has a developing deceleration on the centerline, evidence of significant axial gradients) with that of a Q-vortex, and the stability characteristics of a columnar vortex coinciding with their measured profiles are likely to be very different from those of the Q-vortex. Furthermore, there is good reason to believe that a nonswirling flow would also exhibit an instability at the sharp-edged contraction that forms the entrance to the exit tube; that is, it seems likely that the instability observed is due to the design of the experimental apparatus, and not to the properties of the vortex.

VII. Nonaxisymmetric Motion in S- and B-Breakdown

The theories described in Sec. V make no distinction between S- and B-breakdown, yet there is a clear difference between them. With fixed core radius, the S-breakdown is the low swirl form. This is seen from Fig. 7 (where fixing the Reynolds number effectively fixes the size of the core), and from the data in Table 1, where the Froude number N is seen to be larger for S-breakdowns than for B-breakdowns in a vortex with the same core size. The discrimination between the two forms of breakdown is an important question yet to be addressed by theory.

One essential feature of S-breakdown is the presence of both $|n|=1$ and $n=0$ modes in its Fourier decomposition. The need for the $n=0$ mode was explained in Sec. IV. The $n=1$ modes are needed to explain the corkscrew shape taken on by a streakline released on the vortex axis, since these are the only modes having a nonzero radial velocity component on the axis. Thus, a theoretical treatment of S-breakdown should include these modes as a minimal representation of the velocity field. Furthermore, because of the observations made in the previous section concerning the instability of the wake and the marginal stability of the approach flow in B-form breakdowns, it seems that the same minimal representation is required for both the B- and the S-forms of breakdown.

Because it is believed that the weakly nonlinear axisymmetric wave models have provided some insight into the B-breakdown, similar perturbation ideas will be used here to briefly speculate about the role of the nonaxisymmetric flow features. Suppose a slowly varying axisymmetric basic flow is slightly perturbed and that, to lowest order, the perturbation consists of a superposition of $n=0$ and $|n|=1$ modes, with the axial velocity perturbation given to lowest order by

$$\epsilon A(z,t)\,w_0(r) + \epsilon_s\left[S(z,t)\,w_I(r)\exp\left[i\left(\frac{kz-\omega t}{\Delta}\right)\right]\exp(-i\theta)+\mathrm{cc}\right] \tag{30}$$

where ϵ, ϵ_s, and Δ are small parameters. If $\epsilon_s=0$, the form given describes the long axisymmetric waves discussed in Sec. V; the function S describes the amplitude modulation of the $n=1$ (spiral) modes, and is allowed here to vary slowly compared to the length scale $[0(\Delta/k)]$ and time scale $[0(\Delta/\omega)]$ of these modes. If $\epsilon=0$, then soliton solutions are possible[48] and are described by the nonlinear Schroedinger equation

$$iS_t + \epsilon_s^2\alpha S_{zz} + \epsilon_s^2\beta S\,|S|^2 = 0 \tag{31}$$

The disturbances that emerge when both A and S are present will depend upon how they are coupled, and this will depend in a complex way upon the basic flowfield. To fix ideas, think of the flow in a tube that is first cylindrical, then diverging. Assume infinite Reynolds number, and the existence of a basic axisymmetric flow. Assume, furthermore, that this flow is stable throughout, that it is supercritical at the beginning of the tube divergence, and that at the end, it is critical or only slightly supercritical. This picture is presumably what would be expected in the experiments conducted by Sarpkaya,[30-32] and by the author and co-workers, if vortex breakdown had not occurred.

Now, assume that a weak disturbance is introduced at the downstream end. The linear group velocities of the nonaxisymmetric disturbance modes are positive, under the assumptions adopted, so the nonaxisymmetric features will wash downstream. The linear group velocity of the axially symmetric long wave components of the disturbance will be very small, but the speed of a long wave of finite amplitude may allow upstream propagation. Consider a disturbance that has made its way upstream. We may assume $S=0$ since by assumption infinitesimal $n\neq 0$ modes are washed downstream; thus, an equation like Eq. (15) is operative. As this axisymmetric disturbance moves upstream, it amplifies and its length decreases as a result of the area reduction it sees as it progresses. If the weakly nonlinear Eq. (15) holds throughout, and if dissipation can be ignored in it, then the wave will pass right through the diverging section and continue to propagate upstream.

On the other hand, as its amplitude grows, the stability of the finite amplitude axisymmetric wave to $n\geq 1$ disturbances becomes increasingly suspect. In particular, if the basic flow is not far from marginal, and if the axisymmetric wave destabilizes the flow to $n\geq 1$ modes, then a modest wave amplitude might be expected to lead to growth of three-dimensional disturbances, with linear growth rate proportional to $\epsilon - \epsilon_c$, where ϵ_c is the value of wave amplitude parameter corresponding to marginal instability to nonaxisymmetric perturbations. The presence of a coherent periodic signal in the spectra of disturbances measured by Singh and Uberoi[67] and by Garg and Leibovich[36] suggests that the nonaxisymmetric amplitude $S(z,t)$ grows out of this instability as a Hopf bifurcation.[68] This implies a saturation for the amplitude of the three-dimensional perturbations, with $|\epsilon_s S|^2 = 0(\epsilon - \epsilon_c)$.

The gain in energy of the spiral mode comes at the expense of the axisymmetric mode. With $|S| \neq 0$, there is a feedback into the equation for A of a term of the form $-\gamma |S|^2 A$ where γ is a coupling constant to be found as a functional of the basic flow. Thus, the self-excitation of the S mode in this scenario is made possible by virtue of the presence of the A mode, and the transfer of energy to it from the A mode causes the dissipation of axisymmetric wave energy necessary to stabilize the position of the axisymmetric mode in the diverging part of the channel, since we may take

$$\nu = \gamma |S|^2 \tag{32}$$

in Eq. (15).

Thus, if Eqs. (15) and (32) are adopted, the position and strength of breakdown are determined by the Froude number N of the basic flow, the imposed external forcing (represented in the example by wall tube slope function $g(z)$), and the energy density of the nonaxisymmetric motions, which depends upon how close the basic flow is to marginal instability.

This scenario is not yet supported by detailed calculation, but the conception seems plausible and provides at least a partial unification of the phenomena observed. The factors that distinguish between B-forms and S-forms of breakdown are not yet apparent. Recognizing from Fig. 8 that the S-form appears under more stable and more highly supercritical conditions, it might be conjectured from inviscid theory that the S-form may be unstable at lower values of q, and that the B-form emerges from the S-form as a secondary bifurcation.

VIII. Conclusion

Given a model describing the smooth streamwise evolution of a vortex core, the position of breakdown may be estimated by computing the Froude number N, the stability ratio q/q_c where q_c corresponds to the critical value of q. In the observations that can be documented, stationary vortex breakdown occurs in supercritical regions, $N > 1$. The location seems to correlate better with the point at which the vortex reaches a marginally stable condition than it correlates with the point at which critical conditions are obtained.

A new conceptual framework is suggested based upon information derived from experimental studies of vortex breakdown. In the picture put forward, growth of axisymmetric modes occurs as a result of propagation in a variable vortex waveguide, and this wave loses stability to three-dimensional disturbances when it reaches a critical amplitude. Energy drain from the axisymmetric wave to the nonaxisymmetric modes serves to fix the possible equilibrium positions of the vortex breakdown, which is identified with a large amplitude axisymmetric wave and smaller amplitude superposed nonaxisymmetric wave modes.

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